Dictionary Learning for Directed Graph Signals via Augmented GFT

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Abstract—**This paper proposes a method for designing directed graph (digraph) filters through a dictionary learning approach. Practical digraph filtering methods have not yet been established because of the difficulties posed by asymmetry of the adjacency matrix of a digraph. *Augmented graph Fourier transform* (AuGFT), proposed by Kitamura *et al.,* defines a new graph Laplacian and extends the conventional graph Fourier transform (GFT) for undirected graph signals to directed ones. However, challenges remain in filter design through AuGFT, particularly in determining the skew intensity parameters. Therefore, this study aims to establish a design method for digraph filters with AuGFT. Filters are parameterized with AuGFT, and parameter optimization is performed using a dictionary learning technique. To verify the effectiveness of the proposed method, experimental results of digraph filtering are shown for temperature data of contiguous US and the GSP-traffic-dataset. Compared with undirected graph filtering, the proposed method is shown to have high steerability in designing digraph filters.**

I. Introduction

Graph signal processing (GSP) is theoretical framework for signal processing based on the analysis of the graph spectral properties, where signal domain is defined on the vertices [1], [2]. When the graph structure is known, graph signal processing provides an efficient means of storing, transmitting, and analyzing a variety of graph signals. Graphs can reflect known or estimated connectivity as a prior knowledge in target models. The theory and algorithms of graph signal processing have primarily been designed for undirected graphs, where the connectivity information between vertices is represented by an adjacency matrix. The graph Laplacian, being a graph operator, is a symmetric matrix and can thus be diagonalizable by an orthonormal matrix. Such orthonormal matrices are used as a basis for the graph Fourier transform (GFT). The eigenvalues of graph operators, known as graph frequencies, are non-negative real numbers, allowing the spectrum corresponding to the frequencies to be sorted according to their magnitudes. Consequently, filtering in the spectral domain is realized and utilized as the primary operation in undirected graph signal processing. However, real-world networks, which often include directional information, are difficult to represent using only connectivity information and are instead represented using digraphs. The graph operators of digraphs, however, are generally not symmetric, offering no guarantee that the graph operator can be diagonalized or that its eigenvalues are real [3], [4]. Therefore, unlike with undirected graphs, defining GFT and graph frequency for digraphs poses significant challenges.

To address these challenges, various approaches have been proposed. For example, studies using symmetric components of graph operators [4], [5], studies on minimizing spectral variance [6], studies on constructing new GFT by applying Jordan decomposition [7], [8] or singular value decomposition [9] to the adjacency matrix, and studies on symmetrization of adjacency matrix [10]. However, many of the existing methods are not easy to apply to GSP, which is based on filtering in the spectral domain.

In [11], Furutani *et al.* redefined graph operators for digraphs as Hermitian symmetric matrices by embedding the directional information into the imaginary part of the graph operator. This approach enables the sorting of graph frequency as a real number and realizes filtering in the graph frequency domain. However, a challenge arises with the Hermite-Laplacian filtering technique: it practically produces complex-valued outputs for real-valued inputs.

In order to solve the conventional digraph filtering problem, in the article [12], Kitamura *et al.* proposed the augmented GFT (AuGFT) [1]. AuGFT generalizes the traditional GFT by splitting the graph Laplacian into the symmetric and skew-symmetric components and separately applying the canonical decompositions so that a redundant real GFT is defined for a digraph. AuGFT enables practical real-valued filtering for digraph signals. This generalized method ensures real-valued outputs with filters designed under little constraint. In parallel to this study, Kwak *et al.* proposed to apply polar decomposition to the adjacency matrix and provides a new interpretable filtering technique in the vertex domain [13]. The method in [13] decomposes the operators into product forms, while our proposed method decomposes them into sum forms. Although both of them can realize practical real filters, in this study, we adopt the latter as the base, which can directly utilize existing filters for undirected graph signals.

Despite the advantage of AuGFT, there remain challenges to establish practical procedures for designing filter efficiently. To further advance the method, we propose a novel filter design method employing dictionary learning technique. We demonstrate its effectiveness through comparative analysis with some conventional filters for undirected graph signals.

The organization of this paper is as follows. In Section 2, we review digraph signal processing. In Section 3, we propose a design method of digraph filters using a dictionary

[1] https://github.com/msiplab/AuGFT

learning technique. The effectiveness of our proposed method is validated in Section 4. Finally, Section 5 concludes our contributions.

II. Review of Digraph Signal Processing

In this section, we introduce graph adjacency matrices and the Hermite symmetric graph Laplacian [12], as a graph operator that plays an important role in digraph signal processing. We also introduce the AuGFT based on them. Note here that the definition of AuGFT is generalized from that in the article [12] so that design parameters can be introduced to control the skew intensities.

A. Review of Digraph

Let us assume a finite digraph $\mathcal{G} = (\mathcal{V}, \mathbf{A})$, where $\mathcal{V} \subset \mathbb{N}_0$ is a set of vertices, $\mathbf{A} = (a_{m,n}) \in [0, \infty)^{N\times N}$ is an adjacency matrix, and $a_{m,n}$ is a weight from vertex m to vertex n. If $a_{m,n} = a_{n,m}$ holds for all $(m, n) \in \mathcal{V}\times\mathcal{V}$, then $\mathcal{G}$ is undirected and the adjacency matrix $\mathbf{A}$ is symmetric, i.e., $\mathbf{A}^\mathsf{T} = \mathbf{A}$, where the superscript T denotes the transposition. Unless otherwise stated, $N = |\mathcal{V}| < \infty$ denotes the number of vertices. Since the adjacency matrix $\mathbf{A}$ of a digraph $\mathcal{G}$ is not symmetric, we use *the Hermitian adjacency matrix* $\mathbf{C}$ defined as

$$\mathbf{C} \coloneqq \mathbf{A}_+ + \mathrm{j}\mathbf{A}_-, \tag{1}$$

which is symmteric and reflects the vertice connectivity of digraph where $\mathbf{A}_+ \coloneqq \frac{1}{2}(\mathbf{A} + \mathbf{A}^\mathsf{T})$ and $\mathbf{A}_- \coloneqq \frac{1}{2}(\mathbf{A} - \mathbf{A}^\mathsf{T})$.

B. Para-graph Laplacian

The para-graph Laplacian $\mathbf{L}$ is defined by symmetric real part and skew-symmetric imaginary part as

$$\mathbf{L} = \mathbf{D} - \mathbf{C} = (\mathbf{D} - \mathbf{A}_+) - \mathrm{j}\mathbf{A}_-, \tag{2}$$

where $\mathbf{D}$ is the degree matrix of $\mathbf{C}$ defined as the diagonal matrix having the m-th diagonal element:

$$d_{m,m} \coloneqq \sum_{n\in\mathcal{V}\setminus\{m\}} \sqrt{c_{m,n}c_{n,m}},\ m \in \mathcal{V}. \tag{3}$$

Since $\mathbf{C}$ is Hermitian symmetric, $c_{n,m} = \bar{c}_{m,n}$ and $c_{n,m}c_{m,n} = \bar{c}_{m,n}c_{m,n} = |c_{m,n}|^2$ hold. This leads to $d_{m,m} = \sum_{n\in\mathcal{V}\setminus\{m\}} |c_{m,n}|$. For an undirected graph, $|c_{m,n}| = a_{m,n}$.

We further have a decomposition

$$\mathbf{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\mathsf{T} + \mathrm{j}\mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^\mathsf{T} \tag{4}$$

from canonical forms for real symmetric and real skew-symmetric matrices [14, Corollary 2.5.11(b)], where $\mathbf{U} \in \mathbb{R}^{N\times N}$ is an orthonormal matrix, $\mathbf{Q} \in \mathbb{R}^{N\times r}$ is a matrix consisting of r orthonormal columns,

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_{N-1}) \tag{5}$$

with $\lambda_n \in [0, \infty)$, $\lambda_n \leq \lambda_m$ for $n < m$, and

$$\mathbf{\Sigma} = s_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus s_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots s_{\frac{r}{2}-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6}$$

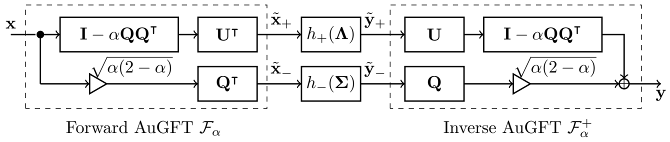


Fig. 1: Digraph filtering $\mathbf{y} = \mathbf{H}\mathbf{x}$, where $\mathbf{U} \in \mathbb{R}^{N\times N}$ and $\mathbf{\Lambda} \in \mathbb{R}^{N\times N}$ are the orthonomal matrix and diagonal matrix obtained by eigenvalue decomposition of $\Re\{\mathbf{L}\}$, and $\mathbf{Q} \in \mathbb{R}^{N\times r}$ and $\mathbf{\Sigma} \in \mathbb{R}^{r\times r}$ are the submatrices of the orthonormal matrix and block diagonal matrix obtained by canonical decomposition of $\Im\{\mathbf{L}\}$, respectively, where $\mathbf{L}$ is a para-graph Laplacian, r is the rank of $\Im\{\mathbf{L}\}$, $h_+(\cdot)$ and $h_-(\cdot)$ are para-graph frequency and skew intensity responses respectively. Note that $r = 0$ for an undirected graph. Therefore, AuGFT reduces to the ordinary GFT. The top channel takes the orthogonal complement space of $\mathrm{span}(\mathbf{Q})$ into account.

with $s_n \in [0, \infty)$, $s_n \geq s_m$ for $n < m$, and r denotes the rank of $\Im\{\mathbf{L}\} = -\mathbf{A}_-$. The symbol $\oplus$ denotes the direct sum of matrices to construct a block diagonal matrix.

$\mathbf{L}$ is a generalization of the conventional graph Laplacian and satisfies the Hermitian symmetric property, i.e., $\mathbf{L}^\mathsf{H} = \mathbf{L}$, and

$$[\mathbf{L}\mathbf{x}]_m = \sum_{n\in\mathcal{V}\setminus\{m\}} |c_{m,n}| \cdot (x_m - e^{\mathrm{j}\angle c_{m,n}} x_n), m \in \mathcal{V}, \tag{7}$$

where the superscript H denotes the Hermitian transposition and $[\cdot]_m$ denotes the m-th element. The para-graph Laplacian quadratic form as a measure of graph signal variation (GSV) is obtained by

$$\Delta_\mathbf{L}(\mathbf{x}) \coloneqq \mathbf{x}^\mathsf{H}\mathbf{L}\mathbf{x} = \sum_{m=0}^{N-1} \sum_{n=m+1}^{N-1} |c_{m,n}| \cdot |x_m - e^{\mathrm{j}\angle c_{m,n}} x_n|^2, \tag{8}$$

where $\mathbf{x} \in \mathbb{C}^N$ (or $\mathbf{x} \in \mathbb{R}^N$).

C. Augmented Graph Fourier transform (AuGFT)

Let $x_n \in \mathbb{R}$ be a signal at vertex n. A digraph signal $\{x_n\}_n$ is denoted by $\mathbf{x} = (x_0, x_1, \cdots, x_{N-1})^\mathsf{T} \in \mathbb{R}^N$. The AuGFT and its inverse are defined by

$$\tilde{\mathbf{x}} = \mathcal{F}_\alpha\mathbf{x} \coloneqq \left(\left(\mathbf{I} - \alpha\mathbf{Q}\mathbf{Q}^\mathsf{T}\right)\mathbf{U} \quad \sqrt{\alpha(2-\alpha)}\mathbf{Q}\right)^\mathsf{T}\mathbf{x}, \tag{9}$$

$$\mathbf{x} = \mathcal{F}_\alpha^+\tilde{\mathbf{x}} \coloneqq \left(\left(\mathbf{I} - \alpha\mathbf{Q}\mathbf{Q}^\mathsf{T}\right)\mathbf{U} \quad \sqrt{\alpha(2-\alpha)}\mathbf{Q}\right)\tilde{\mathbf{x}}, \tag{10}$$

where $\tilde{\mathbf{x}} \in \mathbb{R}^{N+r}$ and $r = \mathrm{rank}(\Im\{\mathbf{L}\})$. $\alpha \in [0, 1]$ is a parameter that controls the intensity of the skew components.

D. Digraph filtering

We realize practical digraph filtering

$$\mathbf{y} = \mathbf{H}\mathbf{x} \tag{11}$$

for digraph signal $\mathbf{x}$, where $\mathbf{y} \in \mathbb{R}^N$ is the filtered digraph signal, and $\mathbf{H} \in \mathbb{R}^{N\times N}$ is a vertex-domain filter through AuGFT. The graph-frequencies $\{\lambda_n\}_n$ and the skew intensities $\{s_n\}_n$ are all real and non-negative.

Fig. 1 shows the procedure of the digraph filtering. The vertex-domain filter $\mathbf{H}$ in (11) is formulated as

$$\mathbf{H} = \mathcal{F}_\alpha^+ \left(h_+(\mathbf{\Lambda}) \oplus h_-(\mathbf{\Sigma}) \right) \mathcal{F}_\alpha, \tag{12}$$

where

$$h_+(\mathbf{\Lambda}) \coloneqq \operatorname{diag}\left(h_+(\lambda_0), h_+(\lambda_1), \cdots, h_+(\lambda_{N-1})\right), \tag{13}$$

$$h_-(\mathbf{\Sigma}) \coloneqq h_-(s_0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus h_-(s_1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots$$
$$\cdots \oplus h_-(s_{\frac{r}{2}-1}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{14}$$

$\mathbf{H}$ is always real if we design both of $h_+(\cdot)$ and $h_-(\cdot)$ to be real, thus the output $\mathbf{y}$ is guaranteed to be real for a real input $\mathbf{x}$.

III. Digraph Dictionary Learning

The innovative AuGFT has made it possible to design digraph filters by decomposes them into responses on graph frequencies $\{\lambda_n\}_n$ and skew intensities $\{s_n\}_n$, however challenges remain in how to design filters or choose parameters, especially in determining the skew intensity responses. In this section, we propose to parameterize filters with α of AuGFT and optimize the parameter with a dictionary learning technique. We consider constructing a P-channel graph filter bank (GFB) $\mathbf{G} = \begin{bmatrix}\mathbf{H}_0 & \mathbf{H}_1 & \cdots & \mathbf{H}_{P-1}\end{bmatrix}$ through dictionary learning. Learning a P-channel GFB from graph signals can be reduced to solving the following optimization problem:

$$\{\hat{\mathbf{G}}, \hat{\mathbf{X}}\} = \arg \min_{\{\mathbf{G},\mathbf{X}\}} \frac{1}{2}\|\mathbf{V} - \mathbf{G}\mathbf{X}\|_\mathrm{F}^2 + \beta\mathscr{R}(\mathbf{X}), \tag{15}$$

where $\mathbf{V}$ is a data matrix of graph signals, $\mathbf{X}$ is a coefficient matrix for $\mathbf{G}$, $\beta \in [0, \infty)$ is the regularization parameter, $\mathscr{R}(\mathbf{X})$ is regularizer imposed to the coefficient matrix and $\|\cdot\|_\mathrm{F}$ denotes the Frobenius norm.

A. Regularizer

Various types of regularizer $\mathscr{R}$ have been studied in the field of signal procesing. For the purpose of dictionary learning, we propose to adopt a typical regularizer, the ℓ_1-norm of the coefficient matrix, as follows

$$\mathscr{R}(\mathbf{X}) = \sum_k \|\mathbf{x}_k\|_1, \tag{16}$$

where $\mathbf{x}_k$ is the k-th column vector of $\mathbf{X}$.

B. Optimization

We consider designing filters through (15) and (16). The minimization problem is alternatively iterated with the following coefficient sparsification problem (Coefficient update) and the dictionary optimization problem (Dictionary update) to obtain optimally learned digraph filters.

- Coefficient update step: given dictionary $\mathbf{G}_{\hat{\theta}}$ in the initial state or trained at the previous step, obtain the optimal coefficients $\hat{\mathbf{X}}$ such that

$$\hat{\mathbf{X}} = \arg \min_{\mathbf{X}} \frac{1}{2}\|\mathbf{V} - \mathbf{G}_{\hat{\theta}}\mathbf{X}\|_\mathrm{F}^2 + \beta \sum_k \|\mathbf{x}_k\|_1, \tag{17}$$

where $\mathbf{G}_{\hat{\theta}}$ is the GFB determined by design parameter $\hat{\theta}$.

- Dictionary update step: given the previous optimal coefficient $\hat{\mathbf{X}}$, learn the design parameter θ of the dictionary such that

$$\hat{\theta} = \arg \min_{\theta} \frac{1}{2}\|\mathbf{V} - \mathbf{G}_\theta\hat{\mathbf{X}}\|_\mathrm{F}^2. \tag{18}$$

IV. Performance Evaluation

In this section, we conducted two experiments involving denoising filter design to evaluate the feasibility of the proposed dictionary learning method. These experiments entailed adding noise $\mathbf{w}$ to a digraph signal $\mathbf{u}$ and subsequently attempting to denoise the observation

$$\mathbf{v} = \mathbf{u} + \mathbf{w} \tag{19}$$

through sparse modeling. The noise is assumed to be additive white Gaussian noise (AWGN) with a standard deviation of σ.

Let $h_{p,+}(\cdot), h_{p,-}(\cdot)$ and α_p be the frequency response, skew response and AuGFT parameter α of the p-th filter $\mathbf{H}_p$, respectively. In the following experiments, $\{h_{p,+}(\cdot)\}$ is determined as a half cosine GFB [15], which is developed for an undirected graph signal, and $\{h_{p,-}(\cdot)\}$ is fixed to a set of the identities, i.e., $h_{p,-}(s) = s$. We control the skewing parameter α_p for each channel independently. That is, $\{\alpha_p\}$ are the design parameters of the following GFBs.

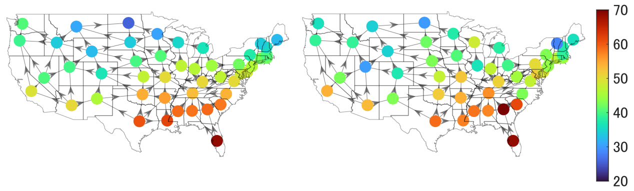
We examine the feasibility of designing filters. To validate the performance of the designed digraph filters, we evaluated the RMSE of the denoised graph signal and the GSV in (8), compared them to the undirected graph case ($\alpha_p = 0$). The coefficient update, involving the ℓ_1-norm regularization, was addressed with the iterative shrinkage-thresholding algorithm (ISTA) for the coefficient update step, and we employed MATLAB's `fmincon` for the dictionary update step. In these experiments, the adjacency matrix weights are set as $a_{m,n} = 1$ for the edge $m \to n$.

A. Experiment for Digraph of Contiguous United States

First, we utilized a digraph with 48 US states (excluding Alaska and Hawaii) as vertices and monthly average temperature data as graph signals for each state. The edges in the graph were connected from lower to higher latitude. Fig. 2 shows the temperature signal on the directed graph of the continuous US used in this experiment. Generally, states closer to the equator, i.e. at lower latitudes, have higher average temperatures. Therefore, we think it is reasonable to assign directed vertices based on the latitude of the state to capture temperature flow. The experimental setup is detailed in Table I.

TABLE I: First experiment setup in IV-A

Number of data for dictionary learning	470
Filter for h_+	Half cosine
Number of filters P	4
Regularization parameter β	0.5
Standard derivation σ	5



(a) (b)

Fig. 2: (a) Graph signal of the average temperature in Fahrenheit of March 2024 in the directed contiguous US. This graph has 48 US states excluding Alaska and Hawaii as vertices. (b) Example of noisy graph signal (RMSE: 5.056).

TABLE II: Cities used for the experiment and their characteristics

City	Country	vertices	Total Edges	Directed Edges
Sakarya	Turkey	103	150	114
Chon Buri	Thailand	108	140	112
Chaoyang (Guangdong)	China	108	206	88
Kanpur	India	117	208	98

B. Experiment for GSP-Traffic-Dataset

Second, we utilized the GSP-Traffic-Dataset[2] [16]. The dataset includes virtual measurements of traffic volumes in 465 cities worldwide, with the graph representing an actual road network with intersections as vertices and roads as edges. Signals in the dataset represent the total number of vehicles passing through intersections over 500 seconds, obtained by simulations. The dataset provides time series data from a 50,000-second simulation.

Given that real roads include one-way streets, the graph is directed, with over 100 vertices in the graph. Consequently, we can conduct the experiments with a graph that has more vertices than the first experiment and, unlike an arborescence, has partially directed edges.

We employed data from four cities for the experiment. The cities used and their characteristics are presented in Table II. We utilized the first 99 data for dictionary learning and the rest for validation. The experimental setup is detailed in Table III.

[2]https://github.com/rukumagai/GSP-Traffic-Dataset

TABLE III: Second experiment setup in IV-B

Filter for h_+	Half cosine
Number of filters P	6
Regularization parameter β	0.2
Standard derivation σ	5

TABLE IV: Average of 100 simulation results in the first experiment of IV-A

	RMSE	GSV
original data	-	93113
noisy data	4.972	96720
undirected graph filtered	4.968	96290
digraph filtered	**4.961**	**95466**

TABLE V: Average of 100 simulation results for each city in the second experiment in IV-B

(a) Sakarya, Turkey

	RMSE	GSV
original data	-	62929
noisy data	5.002	67783
undirected graph filtered	4.816	62280
digraph filtered	**4.810**	**61664**

(b) Chon Buri, Thailand

	RMSE	GSV
original data	-	38553
noisy data	4.999	42936
undirected graph filtered	4.708	37654
digraph filtered	**4.703**	**37323**

(c) Chaoyang, China

	RMSE	GSV
original data	-	50338
noisy data	4.999	56923
undirected graph filtered	4.909	53300
digraph filtered	**4.866**	**51945**

(d) Kanpur, India

	RMSE	GSV
original data	-	45399
noisy data	4.990	51618
undirected graph filtered	4.842	48646
digraph filtered	**4.800**	**47207**

C. Experimental results

The results of the first experiment in IV-A and the second experiment in IV-B are presented in Tables IV and V, respectively. These tables show the average of 100 trial simulations in RMSE and GSV for the original data $\mathbf{u}$, noisy data $\mathbf{v}$, denoised data $\hat{\mathbf{u}}_u$ through undirected graph filters, and denoised data $\hat{\mathbf{u}}_d$ through the proposed digraph filteres.

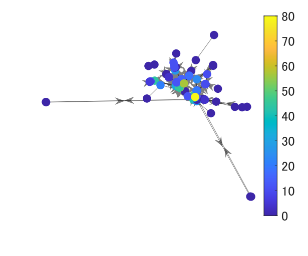
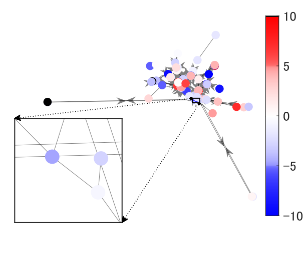
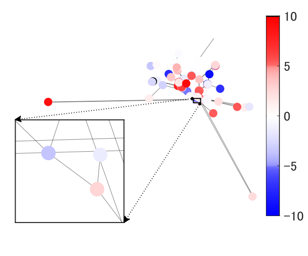
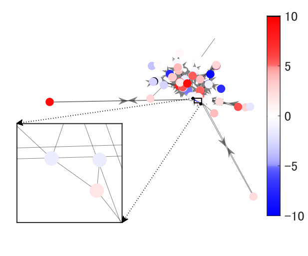
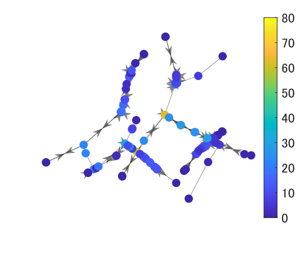
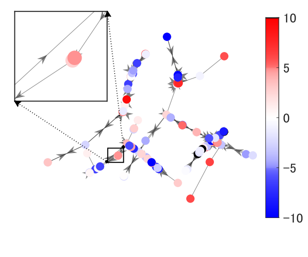
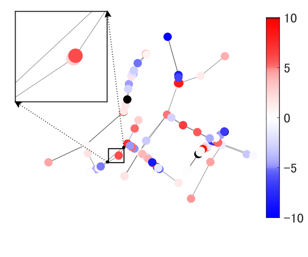
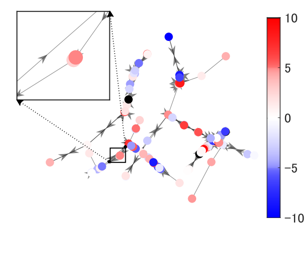
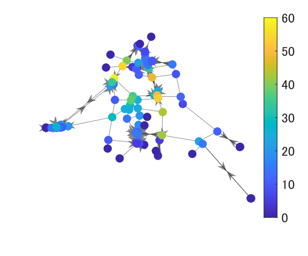
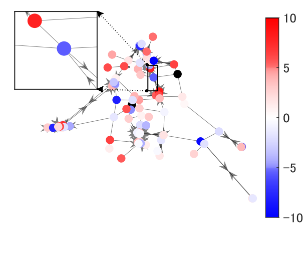
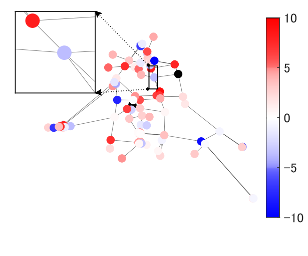
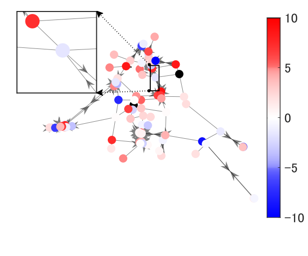
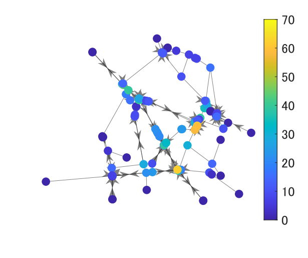
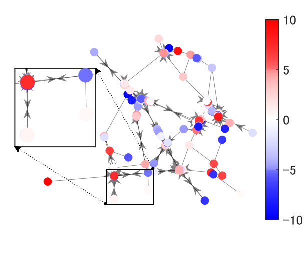
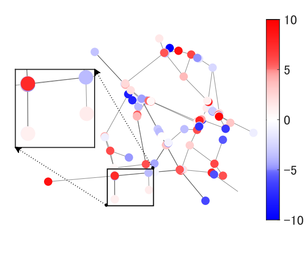
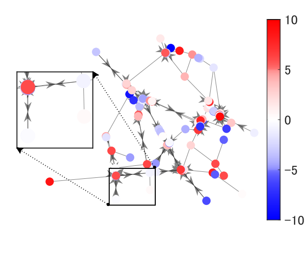
D. Discussion

In both experiments, compared to the undirected graph filters with $\alpha_p = 0$, the RMSEs of our proposal were improved, and the GSV decreased. The results show that digraph denoising filters can be effectively realized by controlling the intensity of the skew components on the graphs. An example of the results from the second experiment in IV-B is shown in Table VI. The graph signals are more smoothed with its adjacent vertex and vertices with directed edges show improvement in the results.

V. Conclusion

In this paper, we proposed a method for designing digraph filters, which ensure real-valued outputs for real-valued inputs, through dictionary learning techniques with AuGFT. We realized the digraph filter design by optimizing GFB parameterized by skewing parameters. The proposed method overcomes the challenges in the conventional digraph signal processing and allows for the design of appropriate filters for digraph signals. The effectiveness of the proposed method was verified through experiments with temperature data of the contiguous US and traffic volume data in four cities provided by GSP-Traffic-Dataset. The introduction of control of the

TABLE VI: Examples of traffic volume graph signal on road for each city. The ‘original’ column gives the graph signal $\mathbf{u}$ used for validation. The ‘noise’ column gives $\mathbf{w} = \mathbf{u} - \mathbf{v}$, the ‘undirected graph filtered’ column gives $\mathbf{u} - \hat{\mathbf{u}}_u$, and the ‘digraph filtered’ column gives $\mathbf{u} - \hat{\mathbf{u}}_d$. The directed edges, expressed by arrows, represent one-way streets, and the undirected edges represent two-way streets.

	original $\mathbf{u}$	noise $\mathbf{w} = \mathbf{u} - \mathbf{v}$	undirected graph filtered $\mathbf{u} - \hat{\mathbf{u}}_u$	directed graph filtered $\mathbf{u} - \hat{\mathbf{u}}_d$
Sakarya, Turkey				
Chon Buri, Thailand				
Chaoyang, China				
Kanpur, India				

skewing parameter confirmed that effective digraph filters can be designed comparing to undirected graph filters. Future work includes further refinement in the filter design method by extending the parameterization in AuGFT domain to design a more flexible GFB and exploring applications to other types of digraph signals.

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References

[1] A. Ortega, *Introduction to Graph Signal Processing*. Cambridge University Press, 2022.

[2] A. Ortega, P. Frossard, J. Kovačević, J. M. F. Moura, and P. Vandergheynst, "Graph signal processing: Overview, challenges, and applications," *Proceedings of the IEEE*, vol. 106, no. 5, pp. 808–828, 2018. DOI: 10.1109/JPROC.2018.2820126.

[3] A. G. Marques, S. Segarra, and G. Mateos, "Signal processing on directed graphs: The role of edge directionality when processing and learning from network data," *IEEE Signal Processing Magazine*, vol. 37, no. 6, pp. 99–116, 2020. DOI: 10.1109/MSP.2020.3014597.

[4] Y. Li and Z.-L. Zhang, "Digraph Laplacian and the Degree of Asymmetry," *Internet Mathematics*, vol. 8, no. 4, pp. 381–401, 2012.

[5] F. Chung, "Laplacians and the cheeger inequality for directed graphs," *Annals of Combinatorics*, vol. 9, no. 1, pp. 1–19, Apr. 2005, ISSN: 0219-3094. DOI: 10.1007/s00026-005-0237-z. [Online]. Available: https://doi.org/10.1007/s00026-005-0237-z.

[6] R. Shafipour, A. Khodabakhsh, G. Mateos, and E. Nikolova, "A directed graph fourier transform with spread frequency components," *IEEE Transactions on Signal Processing*, vol. 67, no. 4, pp. 946–960, 2019. DOI: 10.1109/TSP.2018.2886151.

[7] A. Sandryhaila and J. M. F. Moura, "Discrete signal processing on graphs: Frequency analysis," *IEEE Transactions on Signal Processing*, vol. 62, no. 12, pp. 3042–3054, 2014. DOI: 10.1109/TSP.2014.2321121.

[8] J. Domingos and J. M. F. Moura, "Graph fourier transform: A stable approximation," *IEEE Transactions on Signal Processing*, vol. 68, pp. 4422–4437, 2020. DOI: 10.1109/TSP.2020.3009645.

[9] Y. Chen, C. Cheng, and Q. Sun, "Graph fourier transform based on singular value decomposition of the directed laplacian," *Sampling Theory, Signal Processing, and Data Analysis*, vol. 21, no. 2, p. 24, Aug. 2023, ISSN: 2730-5724. DOI: 10.1007/s43670-023-00062-w. [Online]. Available: https://doi.org/10.1007/s43670-023-00062-w.

[10] V. Satuluri and S. Parthasarathy, "Symmetrizations for clustering directed graphs," Mar. 2011, pp. 343–354. DOI: 10.1145/1951365.1951407.

[11] S. Furutani, T. Shibahara, M. Akiyama, K. Hato, and M. Aida, "Graph signal processing for directed graphs based on the hermitian laplacian," Jun. 2019.

[12] H. Kitamura, H. Yasuda, Y. Tanaka, and S. Muramatsu, "Realization of digraph filters via augmented gft," in *2023 IEEE International Conference on Image Processing (ICIP)*, 2023, pp. 2885–2889. DOI: 10.1109/ICIP49359.2023.10222618.

[13] S. Kwak, L. Shimabukuro, and A. Ortega, "Frequency analysis and filter design for directed graphs with polar decomposition," in *ICASSP 2024 - 2024 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2024, pp. 9661–9665. DOI: 10.1109/ICASSP48485.2024.10446657.

[14] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. Cambridge University Press, 2012.

[15] N. Perraudin, J. Paratte, D. Shuman, *et al.*, "GSPBOX: A toolbox for signal processing on graphs," *ArXiv e-prints*, Aug. 2014. arXiv: 1408.5781 [cs.IT].

[16] R. Kumagai, H. Kojima, H. Higashi, and Y. Tanaka, "Gsp-traffic dataset: Graph signal processing dataset based on traffic simulation," in *Graph Signal Processing Workshop 2024, Delft, The Netherlands*, Jun. 2024.