Optimal Investment With Incomplete Information and Herd Effect

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***Abstract*—Due to incomplete information in financial markets, asset returns and volatility are generally not observable to normal investors. To address this issue, normal investors can use historical price data to estimate these parameters. In this process, they often exhibit bounded rationality, demonstrating their limited cognitive and information-processing capabilities. Also, given the investment expertise of experts, normal investors can refer to and align with the experts' decisions, which is called the herd effect. To better understand normal investors' behaviors, in this work, we modify the general market parameter estimation methods and propose sliding window-based estimation methods to model their bounded rationality. We then formulate the optimal investment problem, which involves one expert and one normal investor whose decisions are influenced by the expert, and jointly consider incomplete information and the herd effect. We derive the analytical solution for the normal investor's optimal decision and theoretically analyze the influence of bounded rationality, incomplete information, and herd effect on his/her decisions. We validate our analysis using numerical experiments on real data.**

I. Introduction

In financial markets, investors allocate assets offering high returns and low volatility to maximize their expected utility of terminal wealth over the investment period [1]. However, the asset returns and volatility are generally not observable for normal investors due to incomplete information [2]. Normal investors can use price data to derive optimal estimates of the returns and volatility [3]. Additionally, with their expertise in accessing and interpreting market information, experts' decisions offer valuable insights into the returns and volatility [4]. When experts anticipate high returns or low volatility, they are more likely to buy the asset, and vice versa. Therefore, normal investors can also refer to and align with the experts' decisions, known as the herd effect [5]. To better understand normal investors' behaviors and devise effective mechanisms to guide their decisions, we first need to study how incomplete information and herd effect influence their decisions.

There exists extensive research on estimating asset returns and volatility with incomplete information. The CAPM [6] and the GARCH model [7] are two classic models for estimating the return and volatility, respectively. The Black-Scholes model simultaneously considers the returns and volatility [8], based on which, the work in [9] provides general methods to jointly estimate the returns and volatility. However, these methods assume that normal investors are fully rational and use all available data for estimation. In reality, normal investors often exhibit bounded rationality with limited cognitive and information-processing capabilities. Due to bounded rationality, they tend to focus on price data within a specific timeframe [10]. To better model their behavior, it is crucial to incorporate bounded rationality into the estimation methods.

Additionally, there have been many qualitative analyses confirming the significant influence of herd effect on investment decisions [11], and our prior work in [12] quantitatively studied the influence of herd effect on the normal investors' decision. However, these studies assume complete information in financial markets and ignore parameter estimation.

In the literature, there are limited works that jointly consider the influence of incomplete information and herd effect on normal investors' decisions. In this work, we address this problem by formulating an optimal investment problem involving one expert and one normal investor. Following the work in [1], we assume that the expert makes rational decisions. For the normal investor, at each time step, we assume that he/she first estimates the asset return and volatility with bounded rationality. Then, he/she makes the optimal decision to maximize the expected utility of the estimated terminal wealth while minimizing the distance between the two investors' decisions due to the herd effect. Finally, the normal investor updates his/her wealth and repeats the above estimation and decision-making process for the next time step.

The rest of this paper is organized as follows. In Section II, we propose sliding window-based market parameter estimation methods considering bounded rationality, and formulate the optimal investment problem for the normal investor, which jointly considers incomplete information and herd effect. In Section III, we derive the analytical solution of the normal investor's optimal decision and quantitatively analyze the influence of bounded rationality, incomplete information, and herd effect on normal investor's decisions. In Section IV, we conduct numerical experiments on a real asset dataset to validate our analysis. Section V is the conclusion.

II. Problem Formulation

In this section, we first define the problem, introduce the expert's rational decision, and extend the general estimation methods for the asset returns and volatility considering the normal investor's bounded rationality. Then, we formulate the optimal investment problem for the normal investor.

The important notations used in this work are in Table I.

A. Problem Definition

Following the work in [1], we consider the financial market with one risk-free asset and one risky asset. Extend-

TABLE I
NOTATIONS

Notation	Meaning	Notation	Meaning
r	Interest rate	μ	Return
v	Excess return	σ	Volatility
α_1, α_2	Risk aversion coefficients	$\mathcal{T}, T$	Investment period
N	Sample size	$\hat{\mu}$	Estimated return
$\hat{v}$	Estimated excess return	$\hat{\sigma}$	Estimated volatility
w	Sliding window width	θ	Herd coefficient
X_1	Wealth	ϕ	Utility function
ρ	Decay rate	D	Average deviation
η	Integral constant	Z_1	Opinion

Notation	Meaning
$\hat{\bar{P}}_1$	Rational decision with incomplete information (*IRD*)
$\bar{P}_1, \bar{P}_2$	Rational decision with complete information (*CRD*)
P_1^*	Optimal decision (*OPD*)

ing this analysis to encompass multiple assets is straightforward. We denote the investment period as $\mathcal{T} := [0, T]$. Let $(\Omega, \mathscr{F}, \{\mathscr{F}(t)\}_{t\in\mathcal{T}}, \mathbb{P})$ be a complete probability space on which a standard Brownian motion $\{B(t)\}_{t\in\mathcal{T}}$ with $B(0) = 0$ is defined. We utilize $\{B(t)\}_{t\in\mathcal{T}}$ to model the randomness of the risky asset price. The market parameters include the interest rate r of the risk-free asset, the return μ, and the volatility σ of the risky asset. Furthermore, we define the excess return of the risky asset over the risk-free asset as

$$v := \mu - r > 0. \tag{1}$$

Typically, investors can easily access the current interest rates from bank counters or websites. Following the work in [1], we assume that both the normal investor and the expert can obtain the true value of r. Compared to the risk-free asset, the risky asset exhibits a higher return and comes with volatility. Following the work in [1], we assume the price process of the risk-free asset $\{S(t)\}_{t\in\mathcal{T}}$ satisfies a geometric Brownian motion (*GBM*), and we assume that the expert can obtain the true values of μ and σ during the whole investment period due to their investment expertise. Conversely, the normal investor needs to estimate μ and σ at each time t from $\{S(u)\}_{u\in[0,t]}$, denoted as $\hat{\mu}(t)$ and $\hat{\sigma}(t)$, respectively.

We use subscripts $i = 1, 2$ to denote the normal investor and the expert, respectively. We define the i-th investor's *decision* $\{P_i(t)\}_{t\in\mathcal{T}}$ as the wealth invested in the risky asset. Following the work in [1], given the true values of r, v, and σ, the expert's rational decision with complete information (*CRD*) $\{\bar{P}_2(t)\}_{t\in\mathcal{T}}$ without considering other's influence is

$$\bar{P}_2(t) := \frac{v}{\alpha_2\sigma^2}\mathrm{e}^{r(t-T)}, t \in \mathcal{T}, \tag{2}$$

where $\alpha_2 > 0$ is called the expert's risk aversion coefficient [13], and a larger α_2 indicates that he/she is less risk-taking.

The normal investor's decision process is in Fig. 1. At each time t, the normal investor with wealth $X_1(t)$ first estimates $\hat{v}(t)$ and $\hat{\sigma}(t)$ using historical price data $\{S(u)\}_{u\in[0,t]}$. Then, he/she makes the optimal decision (*OPD*) $\{P_1^*(u)\}_{u\in[t,T]}$ to maximize the expected utility of the estimated terminal wealth (*EUETW*) $\mathbb{E}_t\phi(\hat{X}_1(T))$ while minimizing the distance between his/her decision and that of the expert $D_t(P_1, \bar{P}_2)$, which we call the average deviation. Finally, he/she invests $P_1^*(t)$ in the risky asset and $X_1(t) - P_1^*(t)$ in the risk-free asset at time t,

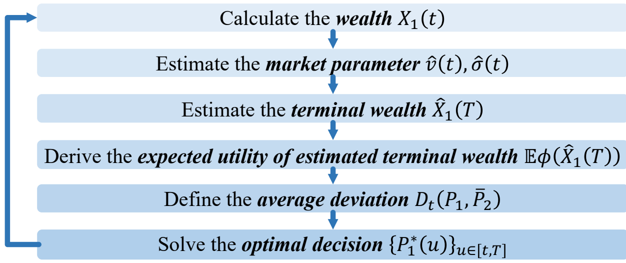


Fig. 1. The normal investor's decision process.

and then proceeds to the next time step $t + \mathrm{d}t$ and updates his/her wealth $X_1(t + \mathrm{d}t)$ using $P_1^*(t)$ and $S(t)$.

B. Market Parameter Estimation Methods

1) The general estimation methods: Based on the statistical properties of *GBM*, the work in [9] provides the following two general estimation methods for the market parameters. To simplify the notation, we denote N as the sample size of the *GBM* and denote the sample value at sample point n as

$$S_n(t) := S\left(\frac{n}{N}t\right), n = 0, \ldots, N. \tag{3}$$

General Method Using Differences (*GEM-diff*): *GEM-diff* is based on the probability distribution of the *difference* between successive observations, i.e., $S_n(t) - S_{n-1}(t), n \in \{1, \ldots, N\}$, and the maximum likelihood estimators (*MLE*s) of $\hat{\mu}_{\text{diff}}^{\text{GEM}}(t)$ and $\hat{\sigma}_{\text{diff}}^{\text{GEM}}(t)$ at time t are

$$\hat{\mu}_{\text{diff}}^{\text{GEM}}(t) = \frac{1}{t}\sum_{n=1}^{N}\frac{S_n(t) - S_{n-1}(t)}{S_{n-1}(t)}, \quad \text{and} \tag{4}$$

$$\hat{\sigma}_{\text{diff}}^{\text{GEM}}(t) = \sqrt{\frac{1}{t}\sum_{n=1}^{N}\left[\frac{S_n(t) - S_{n-1}(t)}{S_{n-1}(t)} - \frac{t}{N}\hat{\mu}_{\text{diff}}^{\text{GEM}}(t)\right]^2}.$$

General Method Using Ratios (*GEM-ratio*): *GEM-ratio* is based on the probability distribution of the ratio between successive observations, i.e., $S_n(t)/S_{n-1}(t), n \in \{1, \ldots, N\}$, and the *MLE*s of $\hat{\mu}_{\text{ratio}}^{\text{GEM}}(t)$ and $\hat{\sigma}_{\text{ratio}}^{\text{GEM}}(t)$ at time t are

$$\hat{\mu}_{\text{ratio}}^{\text{GEM}}(t) = \frac{1}{t}\ln\frac{S(t)}{S(0)} + \frac{[\hat{\sigma}_{\text{ratio}}^{\text{GEM}}(t)]^2}{2}, \quad \text{and} \tag{5}$$

$$\hat{\sigma}_{\text{ratio}}^{\text{GEM}}(t) = \sqrt{\frac{1}{t}\sum_{n=1}^{N}\left[\ln\frac{S_n(t)}{S_{n-1}(t)} - \frac{1}{N}\ln\frac{S(t)}{S(0)}\right]^2}.$$

These two general estimation methods assume that normal investors are fully rational and use *all* historical price data $\{S(t)\}_{t\in\mathcal{T}}$ when estimating μ and r. To model their bounded rationality, we modify these methods and propose the following sliding window-based methods.

2) The sliding window-based methods: From [10], investors often focus on price data within a certain timeframe. Thus, we assume a sliding window with width $w < t$, and the estimators only use $\{S(u)\}_{u\in[t-w,t]}$. We denote the sample value at sample point n with sliding window width w as

$$S_n^w(t) := S\left(\frac{nw}{N} + t - w\right), n = 0, \ldots, N. \tag{6}$$

TABLE II
THE MEAN AND VARIANCE OF DIFFERENT ESTIMATORS

$\hat{\mu}(t)$	$\mathbb{E}\hat{\mu}(t)$	$\mathbb{D}\hat{\mu}(t)$
$\hat{\mu}^{\mathrm{GEM}}_{\mathrm{diff}}(t)$ in (4)	μ	$\frac{\sigma^2}{t}$
$\hat{\mu}^{\mathrm{SWM}}_{\mathrm{diff}}(t)$ in (5)	μ	$\frac{\sigma^2}{w}$
$\hat{\mu}^{\mathrm{GEM}}_{\mathrm{ratio}}(t)$ in (7)	$\mu-\frac{\sigma^2}{2N}$	$\frac{\sigma^2}{t}+\frac{(N-1)\sigma^4}{2N^2}$
$\hat{\mu}^{\mathrm{SWM}}_{\mathrm{ratio}}(t)$ in (8)	$\mu-\frac{\sigma^2}{2N}$	$\frac{\sigma^2}{w}+\frac{(N-1)\sigma^4}{2N^2}$
$\hat{\sigma}^2(t)$	$\mathbb{E}\hat{\sigma}^2(t)$	$\mathbb{D}\hat{\sigma}^2(t)$
$[\hat{\sigma}^{\mathrm{GEM}}_{\mathrm{diff}}(t)]^2$ in (4), $[\hat{\sigma}^{\mathrm{SWM}}_{\mathrm{diff}}(t)]^2$ in (5), $[\hat{\sigma}^{\mathrm{GEM}}_{\mathrm{ratio}}(t)]^2$ in (7), $[\hat{\sigma}^{\mathrm{SWM}}_{\mathrm{ratio}}(t)]^2$ in (8)	$\frac{N-1}{N}\sigma^2$	$\frac{2(N-1)\sigma^4}{N^2}$

Based on *GEM-diff/ratio*, we propose the **Sliding Window-Based Method Using Differences (*SWM-diff*)**, where

$$\hat{\mu}^{\mathrm{SWM}}_{\mathrm{diff}}(t)=\frac{1}{w}\sum_{n=1}^{N}\frac{S^w_n(t)-S^w_{n-1}(t)}{S^w_{n-1}(t)},\quad\text{and}\tag{7}$$

$$\hat{\sigma}^{\mathrm{SWM}}_{\mathrm{diff}}(t)=\sqrt{\frac{1}{w}\sum_{n=1}^{N}\left[\frac{S^w_n(t)-S^w_{n-1}(t)}{S^w_{n-1}(t)}-\frac{w}{N}\hat{\mu}^{\mathrm{SWM}}_{\mathrm{diff}}(t)\right]^2},$$

and the **Sliding Window-Based Method Using Ratios (*SWM-ratio*)**, where

$$\hat{\mu}^{\mathrm{SWM}}_{\mathrm{ratio}}(t)=\frac{1}{w}\ln\frac{S(t)}{S(t-w)}+\frac{[\hat{\sigma}^{\mathrm{SWM}}_{\mathrm{ratio}}(t)]^2}{2},\quad\text{and}\tag{8}$$

$$\hat{\sigma}^{\mathrm{SWM}}_{\mathrm{ratio}}(t)=\sqrt{\frac{1}{w}\sum_{n=1}^{N}\left[\ln\frac{S^w_n(t)}{S^w_{n-1}(t)}-\frac{1}{N}\ln\frac{S(t)}{S(t-w)}\right]^2}.$$

Detailed derivations of (7)–(8) are in the supplementary file.

3) Comparison of different estimation methods: We theoretically analyze the statistical properties of the above estimation methods. The mean and the variance of different estimators, in (4)–(8) are in Table II. We use $\mathbb{E}X$ and $\mathbb{D}X$ to denote the mean and variance of an estimator X. The proofs are in the supplementary file. Note that it is challenging to derive $\mathbb{E}\hat{\sigma}(t)$ and $\mathbb{D}\hat{\sigma}(t)$, so we analyze $\mathbb{E}\hat{\sigma}^2(t)$ and $\mathbb{D}\hat{\sigma}^2(t)$ instead.

From Table II, for the estimation of μ, $\hat{\mu}^{\mathrm{GEM}}_{\mathrm{diff}}(t)$ and $\hat{\mu}^{\mathrm{SWM}}_{\mathrm{diff}}(t)$ are unbiased, while $\hat{\mu}^{\mathrm{GEM}}_{\mathrm{ratio}}(t)$ and $\hat{\mu}^{\mathrm{SWM}}_{\mathrm{ratio}}(t)$ are biased. This suggests that estimating the return using differences gives higher accuracy than ratios. Furthermore, when studying the variance of different estimators, we have $\mathbb{D}\hat{\mu}^{\mathrm{GEM}}_{\mathrm{diff}}(t)<\mathbb{D}\hat{\mu}^{\mathrm{SWM}}_{\mathrm{diff}}(t)$ and $\mathbb{D}\hat{\mu}^{\mathrm{GEM}}_{\mathrm{ratio}}(t)<\mathbb{D}\hat{\mu}^{\mathrm{SWM}}_{\mathrm{ratio}}(t)$, which suggests that estimating the return with bounded rationality results in higher uncertainty compared to the scenario with full rationality.

Additionally, from Table II, for the estimation of σ^2, the mean and the variance of different estimators are the same, which suggests that a bounded rational normal investor's estimation of volatility is as accurate and certain as those of fully rational investors. As the sample size N approaches infinity, $\mathbb{E}\hat{\sigma}^2(t)$ converges to the true value σ^2, and $\mathbb{D}\hat{\sigma}^2(t)$ approaches zero. Therefore, $\hat{\sigma}^2(t)$ consistently converges to σ^2, i.e., $\lim_{N\to\infty}\mathbb{P}(\hat{\sigma}^2(t)=\sigma^2)=1$, and in this case, the normal investor can almost surely obtain the true value of σ^2. In summary, the accuracy and uncertainty of volatility estimation depend on the sample size and are independent of whether the investor is bounded rational or not.

4) Summary: We assume that the expert can acquire the true values of r, μ, v, and σ, while the normal investor can acquire the true value of r, and use (4)–(9) to estimate μ and σ for $t\in\mathcal{T}$. In the following, when not referring to a specific estimator, we simplify the notation by using $\hat{\mu}(t)$ and $\hat{\sigma}(t)$ to denote the estimated parameters in general. We will specify each estimator when comparing their performances. From (1), the *MLE* of the excess return $\hat{v}(t)$ at time t is

$$\hat{v}(t)=\hat{\mu}(t)-r.\tag{9}$$

C. The Normal Investor's Optimal Investment Problem

Next, we analyze the normal investor's wealth, the *EUETW*, and the average deviation, and formulate the problem.

1) The wealth: Following the work in [1], at time t, given the historical price data $\{S(u)\}_{u\in[0,t]}$ and the normal investor's historical *OPD* $\{P_1^*(u)\}_{u\in[0,t]}$, his/her wealth process $\{X_1(u)\}_{u\in[0,t]}$ satisfies

$$\mathrm{d}X_1(u)=r[X_1(u)-P_1^*(u)]\mathrm{d}u+\frac{P_1^*(u)}{S(u)}\mathrm{d}S(u),\tag{10}$$

subject to $X_1(0)=x_1$, where x_1 is his/her initial wealth.

2) The EUETW: Following the work in [1], at time t, given the market parameter estimates $\hat{\mu}(t)$, $\hat{v}(t)$, and $\hat{\sigma}(t)$, the normal investor estimates that his/her wealth $\{\hat{X}_1(u)\}_{u\in[t,T]}$ satisfies

$$\mathrm{d}\hat{X}_1(u)=[r\hat{X}_1(u)+\hat{v}(t)P_1(u)]\mathrm{d}u+\hat{\sigma}(t)P_1(t)\mathrm{d}B(u),\tag{11}$$

subject to $\hat{X}_1(t)=X_1(t)$, if he/she allocates $\{P_1(u)\}_{u\in[t,T]}$ to the risky asset. Based on (11), the normal investor chooses his/her optimal decision $\{P_1^*(u)\}_{u\in[t,T]}$ to maximize his/her *EUETW*. The utility function $\phi(\hat{X}_1(T))$ satisfies the characteristics of the diminishing marginal returns and concavity. Following the work in [13], we consider the following form:

$$\phi(\hat{X}_1(T)):=-\frac{1}{\alpha_1}\mathrm{e}^{-\alpha_1\hat{X}_1(T)},\tag{12}$$

where $\alpha_1>0$ is the normal investor's risk aversion coefficient.

3) The average deviation: Following our prior work in [12], we use the average deviation to measure the distance between the two investors' decisions. At time t, the average deviation between the normal investor's decision $\{P_1(u)\}_{u\in[t,T]}$ and the expert's *CRD* $\{\bar{P}_2(u)\}_{u\in[t,T]}$ is

$$D_t(P_1,\bar{P}_2):=\frac{1}{2}\int_t^T\mathrm{e}^{\rho r(T-u)}[P_1(u)-\bar{P}_2(u)]^2\mathrm{d}u,\tag{13}$$

where $\rho\geqslant 0$ is the decay rate. A larger ρ implies that deviations occurring later carry less weight. When $\rho=0$, deviations for all times are equally weighted.

4) Optimal investment problem: In summary, the optimal investment problem for the normal investor at time t becomes

$$\begin{aligned}\sup_{\mathcal{U}_t\ni\{P_1(u)\}_{u\in[t,T]}}\quad&\mathbb{E}_t\phi(\hat{X}_1(T))-\theta D_t(P_1,\bar{P}_2)\qquad(14)\\ \text{s.t.}\quad&\mathrm{d}\hat{X}_1(u)=[r\hat{X}_1(u)+\hat{v}(t)P_1(u)]\mathrm{d}u+\hat{\sigma}(t)P_1(t)\mathrm{d}B(u),\\ &\mathrm{d}X_1(u)=r[X_1(u)-P_1^*(u)]\mathrm{d}u+\frac{P_1^*(u)}{S(u)}\mathrm{d}S(u),\\ &\hat{X}_1(t)=X_1(t),\quad X_1(0)=x_1,\end{aligned}$$

where $\mathcal{U}_t$ represents the set of admissible decisions, which is a subset of $\mathcal{L}_t^1(\mathcal{T}) := \left\{\{P(u)\}_{u\in[t,T]}\Big|\mathbb{E}\int_t^T |P(u)|\mathrm{d}u < +\infty\right\}$. The herd coefficient $\theta > 0$ is to address the tradeoff between the two different objectives, i.e., maximizing the *EUETW* $\mathbb{E}_t\phi(\hat{X}_1(T))$ and minimizing the average deviation $D_t(P_1, \bar{P}_2)$. Specifically, as θ approaches infinity, the normal investor copies the expert's *CRD*:

$$\lim_{\theta\to+\infty} P_1^*(t) = \bar{P}_2(t), t \in \mathcal{T}. \tag{15}$$

When $\theta = 0$, the normal investor's *OPD* is entirely independent of the expert's decision, and becomes the rational decision using the estimated $\hat{v}(t)$ and $\hat{\sigma}(t)$, i.e.,

$$\hat{\bar{P}}_1(t) := \lim_{\theta\to 0} P_1^*(t) = \frac{\hat{v}(t)}{\alpha_1\hat{\sigma}^2(t)}\mathrm{e}^{r(t-T)}, t \in \mathcal{T}, \tag{16}$$

which we call the normal investor's rational decision with incomplete information (*IRD*). Furthermore, we define the normal investor's *CRD*, i.e., the rational decision using the true values of v and σ as

$$\bar{P}_1(t) := \frac{v}{\alpha_1\sigma^2}\mathrm{e}^{r(t-T)}, t \in \mathcal{T}. \tag{17}$$

From (16) and our prior work in [12], we have

$$\{\hat{\bar{P}}_1(t)\}_{t\in\mathcal{T}} = \mathrm{argmax}_{\mathcal{U}_0\ni\{P_1(t)\}_{t\in\mathcal{T}}}\mathbb{E}_0\phi(\hat{X}_1(T)), \text{ and} \tag{18}$$

$$\{\bar{P}_1(t)\}_{t\in\mathcal{T}} = \mathrm{argmax}_{\mathcal{U}_0\ni\{P_1(t)\}_{t\in\mathcal{T}}}\mathbb{E}_0\phi(X_1(T)). \tag{19}$$

That is, the *IRD* $\{\hat{\bar{P}}_1(t)\}_{t\in\mathcal{T}}$ maximizes the expected utility of the *estimated* terminal wealth $\hat{X}_1(T)$ calculated using (11), while the *CRD* $\{\bar{P}_1(t)\}_{t\in\mathcal{T}}$ maximizes the expected utility of the *real* terminal wealth $X_1(T)$ calculated using (10). From the work in [1], the normal investor's objective is to maximize $\mathbb{E}_0\phi(X_1(T))$. However, due to incomplete information, the normal investor can only obtain $\hat{X}_1(T)$ but not $X_1(T)$ in the decision-making process. From (16) and (17), when the estimators $\hat{v}(t)$ and $\hat{\sigma}(t)$ are more accurate and certain, the *IRD* $\{\hat{\bar{P}}_1(t)\}_{t\in\mathcal{T}}$ aligns better with the *CRD* $\{\bar{P}_1(t)\}_{t\in\mathcal{T}}$, and the normal investor's expected utility of the real terminal wealth $\mathbb{E}_0\phi(X_1(T))$ increases. As mentioned above, the presence of bounded rationality in estimators leads to higher uncertainty, which in turn diminishes the expected utility of the real terminal wealth.

III. Problem Solution and Analysis

In this section, we derive the analytical solution of the normal investor's *OPD* in (14) and theoretically analyze the influence of incomplete information and herd effect on the normal investor's *OPD*. To simplify the notation, we denote

$$\vartheta(t) := \frac{\theta}{\alpha_1\hat{\sigma}^2(t)}, \quad \text{and} \quad \varrho := 2 - \rho \tag{20}$$

as the modified herd coefficient and the modified decay rate, respectively. All proofs are in the supplementary file.

A. The Normal Investor's Optimal Decision

Theorem 1. *The optimal decision $\{P_1^*(t)\}_{t\in\mathcal{T}}$ in* (14) *is*

$$P_1^*(t) = \frac{\eta(t)\hat{v}(t)\mathrm{e}^{(\varrho-1)r(T-t)} + \theta\bar{P}_2(t)}{\eta(t)\alpha_1\hat{\sigma}^2(t)\mathrm{e}^{\varrho r(T-t)} + \theta}, t \in \mathcal{T}, \tag{21}$$

where the integral constant $\eta(t)$ is defined as

$$\eta(t) := -\alpha_1\mathbb{E}_t\phi(\hat{X}_1(T)) = \exp\left\{-\alpha_1 X_1(t)\mathrm{e}^{r(T-t)} - \frac{\hat{v}^2(t)(T-t)}{2\hat{\sigma}^2(t)} + \int_t^T \frac{\vartheta^2(t)\hat{v}^2(t)[\lambda\alpha_1\hat{\sigma}^2(t)/\hat{v}(t) - 1]^2\mathrm{d}u}{2\hat{\sigma}^2(t)[\eta(t)\mathrm{e}^{\varrho r(T-u)} + \vartheta(t)]^2}\right\}, \tag{22}$$

and his/her wealth process $\{X_1(t)\}_{t\in\mathcal{T}}$ is

$$X_1(t) = x_1\mathrm{e}^{rt} + \int_0^t \frac{\eta(u)\hat{v}(u)\mathrm{e}^{\varrho r(T-u)} + \lambda\theta}{\eta(u)\alpha_1\hat{\sigma}^2(u)\mathrm{e}^{\varrho r(T-u)} + \theta}\left(\frac{\mathrm{d}S(u)}{S(u)} - r\mathrm{d}u\right), \tag{23}$$

where the portfolio parameter $\lambda := \frac{v}{\alpha_2\sigma^2} \equiv \bar{P}_2(t)\mathrm{e}^{r(T-t)}$ for $t \in \mathcal{T}$ can be calculated from the expert's CRD [14].

Note that it is challenging to derive the closed-form solution of the normal investor's *OPD* due to the complexity of (21), (22), and (23). To address this problem, we provide a numerical method. We discretize the investment period into several subintervals, each of which has a length of τ, which satisfies that $M := T/\tau$ is an integer, i.e., $\mathcal{T} = \bigcup_{i=1}^M[(i-1)\tau, i\tau]$. Starting from the initial time with a given initial wealth $X_1(0) = x_1$, we first estimate the market parameters $\hat{\mu}(\tau)$ and $\hat{\sigma}(\tau)$, and solve for the integral constant $\eta(\tau)$ using (22). Note that when τ is sufficiently small, we can approximate $X_1(\tau)$ as $X_1(0)$ in (22), i.e.,

$$\eta(\tau) \approx \exp\left\{-\alpha_1 X_1(0)\mathrm{e}^{r(T-\tau)} - \frac{\hat{v}^2(\tau)(T-\tau)}{2\hat{\sigma}^2(\tau)} + \int_\tau^T \frac{\vartheta^2(\tau)\hat{v}^2(\tau)[\lambda\alpha_1\hat{\sigma}^2(\tau)/\hat{v}(\tau) - 1]^2\mathrm{d}u}{2\hat{\sigma}^2(\tau)[\eta(\tau)\mathrm{e}^{\varrho r(T-u)} + \vartheta(\tau)]^2}\right\}. \tag{24}$$

Then, we calculate the *OPD* $P_1^*(\tau)$ and wealth $X_1(\tau)$ using (21) and (23), respectively. We iteratively calculate $\hat{v}(i\tau)$, $\hat{\sigma}(i\tau)$, $\eta(i\tau)$, $P_1^*(i\tau)$, and $X_1(i\tau)$, for $i \in \{2, \ldots, M\}$, until reaching the terminal time T.

B. Influence of Incomplete Information and Herd Effect on the Normal Investor's Optimal Decision

Due to the complexity of (21), it is challenging to theoretically analyze the influence of incomplete information and herd effect on the *OPD*. Note that the normal investor's *IRD* in (16) reflects the influence of incomplete information, while the expert's *CRD* in (2) reflects the influence of herd effect. Therefore, we can decompose the normal investor's *OPD* into his/her *IRD* and the expert's *CRD*, and use the weight function to measure the influence of incomplete information and herd effect on the *OPD*. Following our prior work in [12], the normal investor's *OPD* in (21) is a convex linear combination of his/her *IRD* and the expert's *CRD*.

Theorem 2. *The normal investor's optimal decision* $\{P_1^*(t)\}_{t\in\mathcal{T}}$ *in* (14) *is a convex linear combination of the two investors' rational decisions* $\{\hat{\bar{P}}_1(t)\}_{t\in\mathcal{T}}$ *and* $\{\bar{P}_2(t)\}_{t\in\mathcal{T}}$*, i.e.,*

$$P_1^*(t) = Z_1(t)\hat{\bar{P}}_1(t) + [1 - Z_1(t)]\bar{P}_2(t), t \in \mathcal{T}, \tag{25}$$

where the weight function $\{Z_1(t)\}_{t\in\mathcal{T}}$ *is*

$$Z_1(t) = \frac{\eta(t)\mathrm{e}^{\varrho r(T-t)}}{\eta(t)\mathrm{e}^{\varrho r(T-t)} + \vartheta(t)} \in [0, 1], t \in \mathcal{T}. \tag{26}$$

From (25), $\{Z_1(t)\}_{t\in\mathcal{T}}$ measures the extent to which the normal investor adheres to his/her *IRD*. A larger $\{Z_1(t)\}_{t\in\mathcal{T}}$ indicates that the normal investor's *OPD* aligns more with his/her *IRD*. Conversely, a smaller $\{Z_1(t)\}_{t\in\mathcal{T}}$ indicates a preference for aligning with the expert's *CRD* rather than relying on his/her market parameter estimates when making decisions. Thus, we define $\{Z_1(t)\}_{t\in\mathcal{T}}$ as the normal investor's *opinion* to measure the influence of estimations with incomplete information and herd effect on the *OPD*.

From (20), $\vartheta(t)$ is proportional to θ. A larger θ indicates a higher level of herd effect, and the normal investor is more inclined to the experts' *CRD* when making decisions. In this case, from (26), the opinion $\{Z_1(t)\}_{t\in\mathcal{T}}$ will decrease, and from (25), the normal investor's *OPD* $\{P_1^*(t)\}_{t\in\mathcal{T}}$ will align more with the expert's *CRD* $\{\bar{P}_2(t)\}_{t\in\mathcal{T}}$.

From (22), $\eta(t)$ is negatively proportional to the *EUETW* $\mathbb{E}_t\phi(\hat{X}_1(T))$. Considering the characteristics of the diminishing marginal returns of the utility function, if the *EUETW* $\mathbb{E}_t\phi(\hat{X}_1(T))$ is smaller, i.e., the integral constant $\eta(t)$ is larger, the normal investor is more motivated to increase his/her *EUETW*. In this case, from (26), the opinion $\{Z_1(t)\}_{t\in\mathcal{T}}$ will increase, and thus from (25), the normal investor's *OPD* $\{P_1^*(t)\}_{t\in\mathcal{T}}$ will align more with his/her *IRD* $\{\hat{\bar{P}}_1(t)\}_{t\in\mathcal{T}}$, and he/she is more inclined to estimate the return and volatility himself/herself from historical price data.

IV. Numerical Experiments

In this section, we validate our proposed estimation methods and the analysis of the normal investor's *OPD* in Section II and Section III using numerical experiments.

A. Comparison of Different Estimation Methods

We consider a time frame spanning fifty years from 1974 to 2023, i.e., $T = 50$. We set the true values of the return and the volatility as $\mu = 0.07$ and $\sigma = 0.16$ according to the Dow Jones Industrial Average [15]. We generate 500 trajectories following the *GBM* with $\mu = 0.07$ and $\sigma = 0.16$. We use all the mentioned methods, including *GEM-diff*, *GEM-ratio*, *SWM-diff*, and *SWM-ratio*, to estimate $\hat{\mu}(t)$ and $\hat{\sigma}(t)$, and we compare their performance. We set $N = 1,000$ and $w = 3$, and observe the same trend for other values of N and w.

Fig. 2 shows the estimated return and volatility averaged over $1,000$ simulation results, and we also plot the true values of the return and volatility for comparison. The curves of *SWM-diff* and *SWM-ratio* in Fig. 2 overlap almost completely. From Fig. 2, when $t \in [0, 20)$, the accuracy of parameter estimation across all methods is relatively low due to limited data. Therefore, in the following experiment, we assume that during $t \in [0, 20)$, the normal investor observes data only, while during $t \in [20, T]$, he/she observes data and makes decisions. This assumption also satisfies our problem formulation in Section II, which is explained in the supplementary file.

When $t \in [20, T]$, for the return, the estimators of *GEM-diff/ratio* exhibit lower uncertainty than those of *SWM-diff/ratio*. It indicates that a bounded rational investor has higher uncertainty in estimating the return than a fully rational investor. Additionally, it can be observed that $\hat{\mu}^{\mathrm{GEM}}_{\mathrm{diff}}(t)$ is closer to the true value of μ than $\hat{\mu}^{\mathrm{GEM}}_{\mathrm{ratio}}(t)$. Therefore, estimating the return using differences and with full rationality leads to higher accuracy and certainty than using ratios and with bounded rationality, respectively. For the volatility, the four estimators demonstrate comparable performance. In the supplementary file, we show that as the sample size increases, the estimator of volatility has higher accuracy and certainty. The above results verify our analysis in Section II.

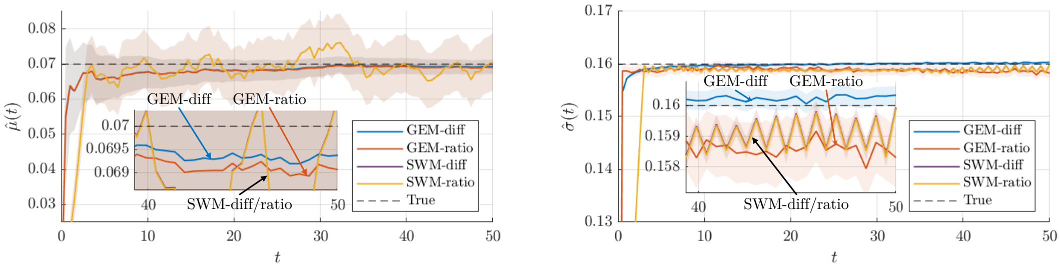
B. Influence of Bounded Rationality on the Normal Investor's Rational Decision and Expected Utility

Next, we simulate the influence of bounded rationality on the *IRD*. We set $T = 50$, $\mu = 0.07$, and $\sigma = 0.16$ as mentioned above. The interest rate r is approximated as 0.04 using the daily average of U.S. 1-Year T-Bills' interest rates in 2022 and 2023 [16]. Following our prior work in [12], we set the normal investor's risk aversion coefficient as $\alpha_1 = 0.2$. Note that the initial wealth can be any positive number [14], and we set $x_1 = 1$ as an example. We set the sliding window width w as 10 and 20. A larger w indicates that the level of bounded rationality is weaker, and thus the estimation method is less uncertain. We observe the same trend for other values of the parameters. We repeat the experiment $1,000$ times. We calculate the *IRD* and the *CRD* using (16) and (17), and their corresponding expected utilities of the real terminal wealth $\mathbb{E}_0\phi(X_1(T))$ using (23) by averaging $\phi(X_1(T))$ in all experiments.

We analyze the influence of bounded rationality on the normal investor's *IRD* in the supplementary file. The experiment results show that when the estimation method is more accurate, the *IRD* aligns better with the *CRD*. Table III shows the expected utility of the real terminal wealth with different estimation methods. From Table III, we can observe that the expected utility with complete information of μ and σ is the highest, and among all estimators, *GEM-diff* achieves the highest expected utility due to its highest accuracy and certainty; while *SWM-ratio* ($w = 10$) gives the lowest expected utility as its accuracy and certainty are the lowest. The above results verify our analysis in Section II.

C. Influence of Incomplete Information and Herd Effect on the Normal Investor's Optimal Decision

Next, we simulate the influence of incomplete information and herd effect on the *OPD*. Apart from the aforementioned parameters, we set the expert's risk aversion coefficient as $\alpha_2 = 0.4$ and the decay rate as $\rho = 2$. We let the herd



(a) The estimated return $\hat{\mu}(t)$

(b) The estimated volatility $\hat{\sigma}(t)$

Fig. 2. Comparison of different estimation methods: the estimated return $\hat{\mu}(t)$ and the estimated volatility $\hat{\sigma}(t)$.

TABLE III

INFLUENCE OF BOUNDED RATIONALITY ON THE NORMAL INVESTOR'S EXPECTED UTILITY OF THE REAL TERMINAL WEALTH

Estimation method	*CRD*	*GEM-diff*	*GEM-ratio*	*SWM-diff* ($w = 20$)	*SWM-ratio* ($w = 20$)	*SWM-diff* ($w = 10$)	*SWM-ratio* ($w = 10$)
$\mathbb{E}_0\phi(X_1(T))$	-3.9137	-4.3601	-4.3643	-4.6937	-4.6950	-6.7842	-6.7980

coefficient θ be 10^{-4}, 10^{-3}, and 10^{-2}. We observe the same trend for other values of the parameters. We assume that the normal investor estimates the market parameters using *GEM-diff*, and calculates the *OPD*s and opinions using Theorem 1 and Theorem 2. The experiment results are in the supplementary file, which show that as the herd coefficient increases, the normal investor's *OPD* aligns more with the expert's *CRD* and deviates further from his/her *IRD*, and his/her opinion decreases, which verifies Theorem 2.

V. CONCLUSION

In this work, we formulate the optimal investment problem jointly considering incomplete information and the herd effect and derive the normal investor's optimal decision. We modify the general market parameter estimation methods and propose the sliding window-based methods considering bounded rationality. Our theoretical analysis shows that while the estimation of the return is more uncertain with bounded rationality, the uncertainty of volatility estimation remains unchanged. As parameter estimation methods become more accurate and certain, the normal investor's rational decision with incomplete information converges to that with complete information, leading to a higher expected utility of real terminal wealth. As the herd effect intensifies, the normal investor's optimal decision aligns more with the expert investor's decision and diverges from his/her own rational decision.

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